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Double-Periodic Boundary Value Problem for Non-linear Dissipative Hyperbolic Equations

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1. INTRODUCTION

Let **Z** and **R** be the set of all integers and real numbers, respectively, and let $\Omega = [0, 2\pi] \times [0, 2\pi]$.

Let $L^1(\Omega)$ be the space of measurable real-valued functions $u: \Omega \to \mathbf{R}$ which are Lebesgue integrable over Ω with usual norm $\|\cdot\|_{L^1}$. Let $L^2(\Omega)$ be the space of measurable real-valued functions $u: \Omega \to \mathbf{R}$ which are Lebesgue square integrable over Ω with usual inner product (,) and usual norm $\|\cdot\|_{L^2}$ and let $L^\infty(\Omega)$ be the space of measurable real-valued functions $u: \Omega \to \mathbf{R}$ which are essentially bounded with the norm

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{(t,x) \in \Omega} |u(t, x)|.$$

Let $C^k(\Omega)$ be the space of all continuous functions $u: \Omega \to \mathbf{R}$ such that the partial derivatives up to order k with respect to both variables are continuous on Ω, while $C(\Omega)$ is used for $C^0(\Omega)$ with the usual norm $\|\cdot\|_\infty$.

Let $W^{k,2}(\Omega)$ be the Sobolev space of all functions $u: \Omega \to \mathbf{R}$ in $L^2(\Omega)$ such that all distributional derivatives $D_t^p D_x^q$ $(0 \leqslant p+q \leqslant k)$ belong to $L^2(\Omega)$ and the norm is given by

$$\|u\|_{W^{k,2}} = \left[\sum_{0 \leqslant p+q \leqslant k} \iint_\Omega [D_t^p D_x^q u(t, x)]^2 \, dt \, dx \right]^{1/2}.$$

In this note, we will investigate the existence of a weak solution of the double-periodic problem for non-linear dissipative hyperbolic equations of the form

$$\beta u_t + u_{tt} - u_{xx} - g(t, x, u) = h(t, x) \qquad \text{in } Q, \tag{1.1}$$

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where $\beta\,(\neq 0)\in \mathbf{R}$, $u=u(t,x)$, $h\in L^2(\Omega)$, and $g\colon \Omega\times\mathbf{R}\to\mathbf{R}$ is a Caratheodory function.

A *weak solution* of the double-periodic problem on Ω for (1.1) will be a $u\in L^\infty(\Omega)$ such that

$$\iint_\Omega u(t,x)[-\beta v_t(t,x)+v_{tt}(t,x)-v_{xx}(t,x)]\,dt\,dx$$
$$+\iint_\Omega g(t,x,u(t,x))\,v(t,x)\,dt\,dx=\iint_\Omega h(t,x)\,v(t,x)\,dt\,dx \tag{1.2}$$

for every $v\in C^2(\Omega)$ satisfying boundary conditions

$$v(t,0)-v(t,2\pi)=v_x(t,0)-v_x(t,2\pi),\qquad t\in[0,2\pi]$$
$$v(0,x)-v(2\pi,x)=v_t(0,x)-v_t(2\pi,x),\qquad x\in[0,2\pi].$$

In our result, we assume $g(t,x,u)$ grows linearly in u; i.e., we assume

(H_1) there exist $a\in L^\infty(\Omega)$ and $b\in L^2(\Omega)$ such that

$$|g(t,x,u)|\leqslant a(t,x)|u|+b(t,x)\qquad \text{a.e. on } \Omega.$$

(H_2) $g(t,x,u)\geqslant 0$ on $\Omega\times\mathbf{R}$.

We will impose more conditions which will be stated below.

In particular, it follows as a corollary of our main result that the double-periodic problem for

$$\beta u_t+u_{tt}-u_{xx}-u^+=h(t,x)\qquad \text{in } \Omega$$

has a weak solution. Here $u^+=\max\{u,0\}$.

Recently several papers have appeared which deal with the double-periodic problem for non-linear dissipative hyperbolic equations. For example, Mawhin [9], Fucik and Mawhin [2], and Drábeck and Lupo [1] investigate the double-periodic problem for non-linear dissipative hyperbolic equations. Mawhin assumes linear growth for the non-linear term g and g is a function of only u. He treats the existence of a weak solution of the double-periodic problem for several types of $g(u)$ in connection with the set $\Sigma=\{k^2-j^2: k,j \text{ integers}\}$. Fucik and Mawhin consider non-linear telegraph equations with non-linear terms of the form

$$g=\mu u^+-\nu u^--\varphi(u),\qquad u=u(t,x),$$

where φ is a continuous and bounded function and μ, ν are real numbers related to the set Σ.

Drábeck and Lupo treat the non-linear dissipative hyperbolic equation when $g = \varphi(t, x, u(t, x), u(t_0, x))$. They assume the function $\varphi(t, x, s, z)$ depends not only on t and x but also on the solution at some fixed time t_0. They allow some bounded oscillations of the function φ over an arbitrary finite number of eigenvalues of the linear part. The author proved the existence and the stability results for a periodic Dirichlet boundary value problem for dissipative hyperbolic equations with superlinear growth nonlinearity in [4, 5], respectively. In our result, we will also allow g to grow at most linearly in u. However, our results have no connection with the so-called Fucik's spectrum of the linear dissipative hyperbolic differential operator and have no boundedness on g. We impose a very weak monotone-like condition on g together with some restrictions at infinity; i.e., $g(t, x, u)$ goes to positive infinity when $u \to +\infty$ and $g(t, x, u)$ approaches zero when $u \to -\infty$.

We make an abstract realization of the linear telegraph differential operator in the double-periodic problem and we get associated abstract operator equations by Fourier series. We also represent the inverse of the linear telegraph operator defined on its range by a convolution product. From this convolution product we have an important norm estimation which is an essential ingredient for our estimates.

Our proof depends on a abstract continuation theorem of Mawhin which we now describe for the convenience of the reader. Now, we introduce Mawhin's theorem as given in [3, 8].

Let $(X, \|\cdot\|_x)$, $(Z, \|\cdot\|_z)$ be real normed spaces. A linear mapping $L\colon \operatorname{Dom} L \subseteq X \to Z$, with $\operatorname{Ker} L = L^{-1}\{0\}$ and $\operatorname{Im} L = L(\operatorname{Dom} L)$, will be called a *Fredholm mapping* if the following two conditions hold:

(a) Ker L has a finite dimension;

(b) Im L is closed and has a finite codimension.

Recall that the codimension of Im L is the dimension of $Z/\operatorname{Im} L$; i.e., the dimension of Coker L of L. When L is a Fredholm mapping, its (Fredholm) *index* is the integer

$$\operatorname{Index} L = \dim[\operatorname{Ker} L] - \operatorname{Codim}[\operatorname{Im} L].$$

We assume throughout that Index $L = 0$.

It follows now from the definition above of a Fredholm mapping and from basic results of linear functional analysis that there exist continuous projections

$$P\colon X \to X \qquad \text{and} \qquad Q\colon Z \to Z$$

such that

$$\operatorname{Im} P = \operatorname{Ker} L \qquad \text{and} \qquad \operatorname{Ker} Q = \operatorname{Im} L$$

so that

$$X = \text{Ker } L \oplus \text{Ker } P \qquad \text{and} \qquad Z = \text{Im } L \oplus \text{Im } Q$$

as topological direct sums. Consequently, the restriction

$$L \mid \text{Dom } L \cap \text{Ker } P \colon \text{Dom } L \to \text{Im } L$$

has an (algebraic) inverse

$$K^R = [L \mid \text{Dom } L \cap \text{Ker } P]^{-1} \colon \text{Im } L \to \text{Dom } L \cap \text{Ker } P.$$

If G is an open bounded subset of X, the mapping N will be caled *L-compact* on $\bar{G}$ if $QN(\bar{G})$ is bounded and $K^R(I-Q)\, N \colon \bar{G} \to X$ is compact.

Im Q is isomorphic to Ker L since Index $L=0$, so there exist isomorphisms

$$J \colon \text{Im } L \to \text{Ker } L.$$

THEOREM 1.1. (Mawhin [8]). *Let L be a Fredholm mapping of index zero and let N be L-compact on G. Suppose*

(a) *For each $\lambda \in \,]0, 1[$, every solution u of*

$$Lu + \lambda Nu = 0$$

is such that $u \notin \partial G$;

(b) *$QNu \neq 0$ for each $u \in \text{Ker } L \cap \partial G$ and the Brouwer degree $d[JQN, G \cap \text{Ker } L, 0] \neq 0$.*

Then the equation $Lu + Nu = 0$ has at least one solution in Dom $L \cap \bar{G}$.

2. PRELIMINARY RESULTS

Now consider the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t, x), \qquad \beta \neq 0 \qquad \text{and} \qquad u = u(t, x)$$

$$u(t, x) = \sum_{(1, m) \in Z \times Z} u_{1m} \exp[i(1t + mx)], \tag{2.1}$$

$$h(t, x) = \sum_{(1, m) \in Z \times Z'} h_{1m} \exp[i(1t + mx)]$$

with $\bar{u}_{1m} = u_{-1-m}$ and $\bar{h}_{1m} = h_{-1-m}$ since u and h are real.

LEMMA 2.1. *$u \in L^2(\Omega)$ is a weak solution if and only if, for all $(1, m) \in Z \times Z$*

$$[\beta 1 i + (m^2 - 1^2)]\, u_{1m} = h_{1m}.$$

Proof. See, e.g., [4].

Let $\mathrm{Dom}\, L = \{u \in L^2(\Omega)$:

$$\sum_{(1, m) \in Z \times Z} [\beta^2 1^2 + (m^2 - 1^2)^2]\, |u_{1m}|^2 < \infty\}.$$

Define an operator $L\colon \mathrm{Dom}\, L \subseteq L^2(\Omega) \to L^2(\Omega)$ by

$$(Lu)(t, x) = \sum_{(1, m) \in Z \times Z} [\beta 1 i + (m^2 - 1^2)]\, u_{1m} \exp[i(1t + mx)].$$

Then it is clear that $\mathrm{Dom}\, L$ is dense in $L^2(\Omega)$, $\mathrm{Ker}\, L = \mathbf{R}$,

$$\mathrm{Im}\, L = \left\{h \in L^2(\Omega) \colon \iint_\Omega h(t, x)\, dt\, dx = 0\right\},$$

$\mathrm{Im}\, L$ is closed,
and

$$[\mathrm{Ker}\, L]^\perp = \mathrm{Im}\, L.$$

Moreover, $L^2(\Omega) = \mathrm{Ker}\, L \oplus \mathrm{Im}\, L$.

Remark 2.1. $L\colon \mathrm{Dom}\, L \subseteq L^2(\Omega) \to L^2(\Omega)$ is a closed operator.

If $h \in L^2(\Omega)$, then u is a weak solution of the double-periodic problem on Ω for the equation

$$\beta u_t + u_{tt} - u_{xx} = h(t, x), \qquad \beta \neq 0 \text{ and } u = u(t, x),$$

if and only if $u \in \mathrm{Dom}\, L$, $Lu = h$ (see, e.g., [7, 10]). L is not bijective but the restriction

$$L|_{\mathrm{Dom}\, L \cap \mathrm{Im}\, L} \colon \mathrm{Im}\, L \cap \mathrm{Dom}\, L \to \mathrm{Im}\, L$$

is bijective, so we can define a right inverse

$$K^R = [L|_{\mathrm{Dom}\, L \cap \mathrm{Im}\, L}]^{-1} \colon \mathrm{Im}\, L \to \mathrm{Im}\, L \cap \mathrm{Dom}\, L$$

and

$$(K^R h)(t, x) = \sum_{\substack{(1, m) \in Z \times Z \\ (1, m) \neq (0, 0)}} [\beta 1 i + (m^2 - 1^2)]^{-1}\, h_{1m} \exp[i(1i + mx)].$$

The following has been already established.

LEMMA 2.2. *Dom* $L \cap \operatorname{Im} L = K^R[\operatorname{Im} L] \subseteq W^{1,2}(\Omega) \cap C(\Omega) \cap \operatorname{Im} L$.

Proof. See, e.g., [2, 9] and Lemma 2.3 in [4].

LEMMA 2.3. *If* $h \in \operatorname{Im} L$, *then there exists a constant* $c > 0$ *independent of* h *such that* $\| K^R h \|_{L^\infty} \leqslant c \, \| h \|_{L^2}$. *The operator* $K^R \colon \operatorname{Im} L \to C(\Omega)$ *is compact.*

Proof. Since

$$(K^R h)(t, x) = \sum_{\substack{(1, m) \in Z \times Z \\ (1, m) \neq (0, 0)}} [\beta 1 i + (m^2 - 1^2)]^{-1} h_{1m} \exp[i(1t + mx)]$$

and since $h_{1m} = (1/4\pi^2) \iint_\Omega h(s, y) \exp[-i(1s + my)] \, ds \, dy$, we can represent K^R as a convolution product $K^R h = K * h$ where

$$K(t, x) = \frac{1}{4\pi^2} \sum_{\substack{(1, m) \in Z \times Z \\ (1, m) \neq (0, 0)}} [\beta 1 i + (m^2 - 1^2)]^{-1} \exp[i(1t + mx)];$$

i.e.,

$$\begin{aligned} (K^R h)(t, x) &= (K * h)(t, x) \\ &= \iint_\Omega K(t - s, x - y) \, h(s, y) \, ds \, dy. \end{aligned}$$

In a way similar to the proof of Lemma 2.2 in [4], we can show that K^R is compact.

Now we can extend K^R to $L^1(\Omega)$ by defining $\bar{K}^R \colon L^1(\Omega) \to L^2(\Omega)$ by the formula

$$(\bar{K}^R h)(t, x) = \iint_\Omega K(t - s, x - y) \, h(s, y) \, ds \, dy \qquad \text{for} \quad h \in L^1(\Omega).$$

Then we have the following lemma.

LEMMA 2.4. $\| \bar{K}^R h \|_{L^2} \leqslant \| K \|_{L^2} \, \| h \|_{L^1}$.

Proof. Since

$$(\bar{K}^R h)(t, x) = \iint_\Omega K(t - s, x - y) \, h(s, y) \, ds \, dy,$$

we have, by extending $h(t, x)$ 2π-periodically in both variables to $\mathbf{R} \times \mathbf{R}$ and then changing variables,

$$(\bar{K}^R h)(t, x) = \int_{t-2\pi}^{t} \int_{x-2\pi}^{x} h(t-u, x-v)\, K(u, v)\, du\, dv.$$

By Hölder's inequality, we have

$$\begin{aligned} |(\bar{K}^R h)(t, x)|^2 &\leqslant \left[\iint_\Omega |h(t-u, x-v)|\, |K(u, v)|\, du\, dv \right]^2 \\ &= \left[\iint_\Omega |h(t-u, x-v)|\, du\, dv \right] \\ &\quad \times \left[\iint_\Omega |h(t-u, x-v)|\, |K(u, v)|^2\, du\, dv \right] \\ &= \|h\|_{L^1} \iint_\Omega |h(t-u, x-v)|\, |K(u, v)|^2\, du\, dv. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} &\iint_\Omega \iint_\Omega |h(t-u, x-v)|\, |K(u, v)|^2\, du\, dv\, dt\, dx \\ &\quad = \iint_\Omega |K(u, v)|^2 \left[\iint_\Omega |h(t-u, x-v)|\, dt\, dx \right] du\, dv \\ &\quad = \|h\|_{L^1} \iint_\Omega |K(u, v)|^2\, du\, dv \\ &\quad = \|h\|_{L^1} \|K\|_{L^2}^2. \end{aligned}$$

Therefore

$$\|\bar{K}^R h\|_{L^2} \leqslant \|h\|_{L^1} \|K\|_{L^2}.$$

Remark 2.2. It is useful to note that if $h \in L^2(\Omega)$, then $\|\bar{K}^R h\|_{L^\infty} \leqslant C_1 \|h\|_{L^2}$ for some constant $C_1 > 0$.

3. Main Results

We now state and prove the main result of this note. Let $g: \Omega \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function. That is, $g(\cdot, \cdot, u)$ is measurable on Ω for each $u \in \mathbf{R}$ and $g(t, x, \cdot)$ is continuous on $\mathbf{R}$ a.e. on Ω. Moreover we assume the following conditions.

(H_3) There exists a non-negative function $\varphi \in L^2(\Omega)$ such that

$$g(t, x, u) \leqslant \varphi(t, x) \qquad \text{a.e. on } \Omega \text{ for } u \leqslant 0.$$

(H_4) $\liminf_{u \to +\infty} g(t, x, u) = \infty$ uniformly on Ω and $\limsup_{u \to -\infty} g(t, x, u) = 0$ for all $(t, x) \in \Omega$.

(H_5) g tends to be non-decreasing in u; i.e., there exist a function $r \in L^2(\Omega)$ and a constant $r_0 \geqslant 0$ such that

$$g(t, x, u_2) \leqslant g(t, x, u_1 + u_2) + r(t, x)$$

holds a.e. on Ω whenever $u_1 \geqslant r_0$.

THEOREM 3.1. *Assume* (H_1)–(H_5). *Then the double-periodic problem on* Ω *for Eq.* (1.1) *has at least one weak solution if* $\iint_\Omega h(t, x)\, dt\, dx < 0$.

Proof. Let $L\colon \operatorname{Dom} L \subseteq L^2(\Omega) \to L^2(\Omega)$ be defined as in Lemma 2.1. Then we know that $\operatorname{Ker} L = \mathbf{R}$, $\operatorname{Im} L = \{h \in L^2(\Omega) : \iint_\Omega h(t, x)\, dt\, dx = 0\}$, $\operatorname{Im} L$ is closed, and $[\operatorname{Ker} L]^\perp = \operatorname{Im} L$. Now consider a continuous projection

$$P\colon L^2(\Omega) \to L^2(\Omega) \qquad \text{such that} \quad \operatorname{Im} L = \operatorname{Ker} P.$$

Then it is easy to see that $L^2(\Omega) = \operatorname{Ker} L \oplus \operatorname{Ker} P$. We consider another continuous projection $Q\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$(Qh)(t, x) = \frac{1}{4\pi^2} \iint_\Omega h(t, x)\, dt\, dx.$$

Then we have $L^2(\Omega) = \operatorname{Im} Q \oplus \operatorname{Im} L$, $\operatorname{Ker} Q = \operatorname{Im} L$, and $L^2(\Omega)/\operatorname{Im} L$ is isomorphic to $\operatorname{Im} Q$.

Since $\dim [L^2(\Omega)/\operatorname{Im} L] = \dim [\operatorname{Im} Q] = \dim [\operatorname{Ker} L] = 1$, we have an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$ and L is a Fredholm mapping of index 0. The right inverse $K^R = [L|_{\operatorname{Dom} L \cap \operatorname{Im} L}]^{-1}\colon \operatorname{Im} L \to L^2(\Omega)$ is compact, since the inclusion mapping $i\colon C(\Omega) \to L^2(\Omega)$ is continuous. Now we define a substitution operator $N\colon L^2(\Omega) \to L^2(\Omega)$ by $(Nu)(t, x) = -g(t, x, u) - h(t, x)$ for $u \in L^2(\Omega)$ and $(t, x) \in \Omega$.

By Krasnosel's kii's results (see, e.g., [6]), N is continuous and bounded. Let G be any open bounded subset of $L^2(\Omega)$. Then it is clear that $QN\colon \bar{G} \to L^2(\Omega)$ is bounded since Qu is the mean value of u, and the operator $K^R(I - Q)\colon \bar{G} \to L^2(\Omega)$ is compact and continuous.

Therefore, N is L-compact on $\bar{G}$.

Now it is easy to check that $u \in L^2(\Omega)$ is a weak solution to the double-periodic problem of (1.1) if and only if $u \in \operatorname{Dom} L$ and

$$Lu + Nu = 0 \qquad \text{(see, e.g., [7])}. \tag{3.1}$$

Since L is a Fredholm mapping of index zero and N is L-compact, by Mawhin's continuation theorem if there exists a bounded open set G in $L^2(\Omega)$ such that

(a) for each $\lambda \in]0, 1[$, every solution u of

$$Lu + \lambda Nu = 0$$

is such that $u \notin \partial G$,

(b) $QNu \neq 0$ for each $u \in \operatorname{Ker} L \cap \partial G$ and

$$d[JQN, G \cap \operatorname{Ker} L, 0] \neq 0,$$

where d is the Brouwer topological degree,

then the equation $Lu + Nu = 0$ has at least one solution in $\operatorname{Dom} L \cap \bar{G}$.

Now we prove (a). For this purpose, let (u, λ) be any solution to

$$Lu + \lambda Nu = 0. \tag{3.2}$$

Then (u, λ) is a weak solution for the double-periodic problem of the equation

$$\beta u_t + u_{tt} - u_{xx} - \lambda g(t, x, u) = \lambda h(t, x) \qquad \text{in } \Omega.$$

Since $u \in \operatorname{Dom} L \subseteq L^2(\Omega)$ and $L^2(\Omega) = \operatorname{Ker} L \oplus \operatorname{Im} L$, $u = \alpha + u_1$ where $\alpha \in \operatorname{Ker} L$ and $u_1 \in \operatorname{Im} L$.

By applying $\bar{K}^R$ on both sides of Eq. (3.2), we have, since $\bar{K}^R|_{\operatorname{Im} L} = K^R$,

$$\bar{K}^R L u_1 + \lambda \bar{K}^R Nu = 0,$$

or

$$u_1 = -\lambda \bar{K}^R Nu = -\lambda[\bar{K}^R \bar{g}(u) + \bar{K}^R h],$$

where $\bar{g}(u)(t, x) = g(t, x, u(t, x))$. Since

$$\|K\|_{L^2} = \sum_{\substack{(1, m) \in Z \times Z \\ (1, m) \neq (0, 0)}} [\beta^2 1^2 + (m^2 - 1^2)^2]^{-1} < \infty,$$

$$\|u_1\|_{L^2} \leqslant \|K\|_{L^2} [\|\bar{g}(u)\|_{L^1} + \|h\|_{L^1}].$$

By taking the inner product with 1 on both sides of (3.2), since $1 \in \operatorname{Ker} L$, $(Nu, 1) = 0$, or equivalently,

$$-\iint_\Omega g(t, x, u(t, x))\, dt\, dx = \iint_\Omega h(t, v)\, dt\, dx.$$

Therefore, $\|\bar{g}(u)\|_{L^1} = \|h\|_{L^1} \leqslant C2\pi \|h\|_{L^2}$. Hence $\|u_1\|_{L^2} \leqslant 4\pi C \|K\|_{L^2} \|h\|_{L^2}$. So $\|u_1\|_{L^2} \leqslant M$ for some $M > 0$ independently of $\lambda \in]0, 1[$.

We will show that for any solution $(u, \lambda) = (\alpha + u_1, \lambda)$ is also bounded independently of $\lambda \in]0, 1[$. Suppose that there exists a sequence $\{(u_n, \lambda_n)\}$ of solutions with $u_n = \alpha_n + u_{1n}$ and $\{|\alpha_n|\}$ is unbounded. Then, there exists a subsequence, say again $\{\alpha_n\}$, such that $\alpha_n \to +\infty$ or $\alpha_n \to -\infty$ as $n \to +\infty$. Since $\|u_{1n}\|_{L^2} \leqslant M$, $\|u_{1n}\|_{L^1} \leqslant M_0$ where $M_0 = 2\pi M_1$. Now let $G(M_0, n) = \{(t, x) : u_{1n}(t, x) \leqslant -(M_0 + 1)/m(\Omega)\}$. Then

$$\begin{aligned} M_0 &\geqslant \iint_\Omega |u_{1n}| \, dt \, dx \\ &\geqslant \iint_{G(M_0, n)} |u_{1n}| \, dt \, dx \\ &\geqslant \iint_{G(M_0, n)} \frac{M_0 + 1}{m(\Omega)} \, dt \, dx \\ &= \frac{M_0 + 1}{m(\Omega)} m[G(M_0, n)]. \end{aligned}$$

Therefore, $(M_0/(M_0 + 1)) \, m(\Omega) \geqslant m[G(M_0, n)]$. So if $S(M_0, n) = \{(t, x) : u_{1n}(t, x) \geqslant -(M_0 + 1)/m(\Omega)\}$, then

$$\begin{aligned} m[S(M_0, n)] &\geqslant m(\Omega) - \frac{M_0}{M_0 + 1} m(\Omega) \\ &= \left(1 - \frac{M_0}{M_0 + 1}\right) m(\Omega), \end{aligned}$$

Hence $m[S(M_0, n)] \geqslant \delta_0 > 0$ where

$$\delta_0 = \left(1 - \frac{M_0}{M_0 + 1}\right) m(\Omega).$$

Since $\liminf_{u \to +\infty} g(t, x, u) = \infty$ uniformly on Ω, for

$$\varepsilon = \frac{-\iint_\Omega h(t, x) \, dt \, dx + 2\pi \|\varphi\|_{L^2} + 1}{\delta_0} > 0,$$

there exists a constant $C_2 > 0$ such that

$$g(t, x, u) \geqslant \varepsilon \text{ uniformly on } \Omega \qquad \text{if} \quad u \geqslant C_2.$$

Since $\alpha_n \to +\infty$, there exists an $N > 0$ such that

$$\alpha_n \geqslant \frac{M_0 + 1}{m(\Omega)} + C_2 \qquad \text{if} \quad n \geqslant N.$$

Hence, for $(t, x) \in S(M_0, n)$, $n \geqslant N$, we have

$$\begin{aligned} u_n(t, x) &= \alpha_n + u_{1n}(t, x) \\ &\geqslant \frac{M_0 + 1}{m(\Omega)} + C_2 - \frac{M_0 + 1}{m(\Omega)} \\ &= C_2 > 0. \end{aligned}$$

Therefore, $g(t, x, u_n(t, x)) \geqslant \varepsilon$ on $S(M_0, n)$ if $n \geqslant N$. So, for $n \geqslant N$,

$$\begin{aligned} &-\iint_\Omega h(t, x)\, dt\, dx \\ &\quad = \iint_\Omega g(t, x, u_n(t, x))\, dt\, dx \\ &\quad = \iint_{u_n(t, x) > 0} g(t, x, u_n(t, x))\, dt\, dx + \iint_{u_n(t, x) \leqslant 0} g(t, x, u_n(t, x))\, dt\, dx \\ &\quad \geqslant \iint_{S(M_0, n)} g(t, x, u_n(t, x))\, dt\, dx - \iint_{u_n(t, x) \leqslant 0} |\, g(t, x, u_n(t, x))|\, dt\, dx \\ &\quad \geqslant \iint_{S(M_0, n)} \epsilon\ dt\, dx - \iint_{u_n(t, x) \leqslant 0} |\, \varphi(t, x)|\, dt\, dx \\ &\quad \geqslant \varepsilon m[S(M_0, n)] - \iint_\Omega |\, \varphi(t, x)|\, dt\, dx \\ &\quad \geqslant \varepsilon\delta_0 - 2\pi\, \|\varphi\|_{L^2} \\ &\quad = \frac{-\iint_\Omega h(t, x)\, dt\, dx + 2\pi\, \|\varphi\|_{L^2} + 1}{\delta_0}\, \delta_0 - 2\pi\, \|\varphi\|_{L^2} \\ &\quad = -\iint_\Omega h(t, x)\, dt\, dx + 1 \end{aligned}$$

which is impossible. Therefore, there exists a constant $K_0 > 0$ such that $\alpha_n \leqslant K_0$ for all $n = 1, 2, \ldots$.

Since $\{(u_n, \lambda_n)\}$ is a sequence of weak solutions, we have

$$Lu_n + \lambda_n N u_n = 0, \qquad u_n = \alpha_n + u_{1n}, \qquad \text{and} \quad n = 1, 2, 3, \ldots . \tag{3.3}$$

Now consider the following two cases:

Case I. Suppose $u_n = \alpha_n + u_{1n}$ with $\alpha_n \geqslant 0$. Then $0 \leqslant \alpha_n \leqslant K_0$ and, by applying $\bar{K}^R$ on each side of (3.3), we have, by Remark 2.2,

$$\begin{aligned}
\| u_{1n} \|_{L^\infty} &\leqslant C_1 \lambda_n \| N u_n \|_{L^2} \\
&\leqslant C_1 [\| g(\cdot, \cdot, u_n(\cdot, \cdot)) \|_{L^2} + \| h \|_{L^2}] \\
&\leqslant C_1 \left\{ \iint_\Omega | g(t, x, \alpha_n + u_{1n}(t, x))|^2 \, dt \, dx \right\}^{1/2} + C_1 \| h \|_{L^2} \\
&\leqslant C_1 \left\{ \iint_\Omega [| a(t, x)|(|\alpha_n| + |u_{1n}|) + |b(t, x)|]^2 \, dt \, dx \right\}^{1/2} + C_1 \| h \|_{L^2} \\
&\leqslant C_1 \{ \| a \|_{L^\infty} (K_0^2 + 2K_0 M_0 + M^2 + 2K_0 \| b \|_{L^1} + M \| b \|_{L^2} \\
&\quad + \| b \|_{L^2}) \}^{1/2} + C_1 \| h \|_{L^2}.
\end{aligned}$$

Therefore, $\| u_{1n} \|_{L^\infty} \leqslant C_2$ for some $C_2 > 0$ if $\alpha_n \geqslant 0$.

Case II. Suppose $u_n = \alpha_n + u_{1n}$ and $\alpha_n \leqslant 0$. For $n \geqslant 0$ such that $-r_0 \leqslant \alpha_n \leqslant 0$, we have

$$\begin{aligned}
&\| u_{1n} \|_{L^\infty} \\
&\quad \leqslant C_1 \| g(\cdot, \cdot, u(\cdot, \cdot)) \|_{L^2} + C_1 \| h \|_{L^2} \\
&\quad = C_1 \left\{ \iint_\Omega | g(t, x, u_n(t, x))|^2 \, dt \, dx \right\}^{1/2} + C_1 \| h \|_{L^2} \\
&\quad \leqslant C_1 \left\{ \iint_\Omega [| a(t, x)| \, | u_n(t, x)| + | b(t, x)|]^2 \right\}^{1/2} + C_1 \| h \|_{L^2} \\
&\quad \leqslant C_1 \left\{ \iint_\Omega [| a(t, x)|(|\alpha_n| + | u_{1n}(t, x)|) + | b(t, x)|]^2 \right\}^{1/2} \\
&\qquad + C_1 \| h \|_{L^2} \\
&\quad \leqslant C_1 \left\{ \iint_\Omega [| a(t, x)|(r_0 + | u_{1n}(t, x)|) + | b(t, x)|]^2 \right\}^{1/2} \\
&\qquad + C_1 \| h \|_{L^2}
\end{aligned}$$

$$\leqslant C_1 \left\{ \| a \|_{L^\infty}^2 \left[4\pi^2 r_0^2 + 2r_0 \iint_\Omega |u_{ln}(t, x)| \, dt \, dx \right.\right.$$

$$\left. + \iint_\Omega |u_{ln}(t, x)|^2 \, dt \, dx \right]$$

$$+ 2 \| a \|_{L^\infty} \left(r_0 \iint_\Omega |b(t, x)| \, dt \, dx \right.$$

$$\left. + \iint_\Omega |u_{1n}(t, x)| \, |b_{1n}(t, x)| \, dt \, dx \right)$$

$$\left. + \iint_\Omega |b(t, x)|^2 \, dt \, dx \right\}^{1/2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \{ \| a \|_{L^\infty}^2 (4\pi^2 r_0^2 + 4 r_0 \pi \| u_{1n} \|_{L^2} + \| u_{1n} \|_{L^2}^2)$$

$$+ 2 \| a \|_{L^\infty} (2\pi r_0 \| b \|_{L^2} + \| u_{1n} \|_{L^2} \| b \|_{L^2})$$

$$+ \| b \|_{L^2}^2 \}^{1/2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \{ \| a \|_{L^\infty}^2 (4\pi^2 r_0^2 + 4 r_0 \pi M + M^2)$$

$$+ 2 \| a \|_{L^\infty} \| b \|_{L^2} (2\pi r_0 + M) + \| b \|_{L^2}^2 \}^{1/2} + C_1 \| h \|_{L^2}.$$

Therefore, there exists a constant $C_3 > 0$ such that $\| u_{1n} \|_{L^\infty} \leqslant C_3$. Now, for $n \geqslant 0$ such that $\alpha_n \leqslant -r_0$, we have

$$\| u_{1n} \|_{L^\infty}$$

$$\leqslant C_1 \| g(\cdot, \cdot, u_n(\cdot, \cdot)) \|_{L^2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \left\{ \iint_\Omega | g(t, x, u_n(t, x))|^2 \, dt \, dx \right\}^{1/2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \left\{ \iint_\Omega (| g(t, x, u_{1n}(t, x))| + |r(t, x)|)^2 \right\}^{1/2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \left\{ \iint_\Omega (|a(t, x)| \, |u_{1n}(t, x)| + |b(t, x)| + |r(t, x)|)^2 \right\}^{1/2}$$

$$+ C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \{ \| a \|_{L^\infty}^2 (\| u_{1n} \|_{L^2}^2 + \| b \|_{L^2}^2 + \| r \|_{L^2}^2)$$

$$+ 2 \| a \|_{L^\infty} \| u_{1n} \|_{L^2} (\| b \|_{L^2} + \| r \|_{L^2})$$

$$+ 2 \| b \|_{L^2} \| r \|_{L^2} \}^{1/2} + C_1 \| h \|_{L^2}$$

$$\leqslant C_1 \{ \| a \|_{L^\infty}^2 (M^2 + \| b \|_{L^2}^2 + \| r \|_{L^2}^2)$$

$$+ 2 \| a \|_{L^\infty} \| u_{1n} \|_{L^2} (\| b \|_{L^2} + \| r \|_{L^2})$$

$$+ 2 \| b \|_{L^2} \| r \|_{L^2} \}^{1/2} + C_1 \| h \|_{L^2}.$$

Hence, there exists a constant $C_4 > 0$ such that $\| u_{1n} \|_{L^\infty} \leqslant C_4$ for $n \geqslant 0$ such that $\alpha_n \leqslant -r_0$. Therefore, $\| u_{1n} \|_{L^\infty} \leqslant C_5 = \max\{C_3, C_4\}$ for $n \geqslant 0$ if $\alpha_n \leqslant 0$. Hence $\| u_{1n} \|_{L^\infty} \leqslant C_0$ for some C_0, for all $n = 1, 2, 3, \ldots$.

Now we know $\alpha_n \leqslant K_0$. If $\alpha_n \to -\infty$ as $n \to +\infty$, then $u_n(t, x) = \alpha_n + u_{1n}(t, x) \leqslant \alpha_n + \| u_{1n} \|_{L^\infty} \leqslant \alpha_n + C_0$ and $u_n(t, x) \to -\infty$ as $n \to +\infty$.

Since $\alpha_n \to -\infty$ as $n \to +\infty$, there exists $N > 0$ such that $\alpha_n \leqslant -C_0$ for $n \geqslant N$. Therefore $u_n(t, x) \leqslant 0$ a.e. if $n \geqslant N$. Hence, by (H_3), we have

$$g(t, x, u_n(t, x)) \leqslant \varphi(t, x) \qquad \text{a.e.} \qquad \text{if} \quad n \geqslant N.$$

So $\limsup_{n \to +\infty} g(t, x, u_n(t, x)) = 0$ a.e. Thus, for $n \geqslant N$, we have

$$-\iint_\Omega h(t, x)\, dt\, dx = \iint_\Omega g(t, x, u_n(t, x))\, dt\, dx$$

and hence

$$\begin{aligned} -\iint_\Omega h(t, x)\, dt\, dx &= \limsup_{n \to -\infty} \iint_\Omega g(t, x, u_n(t, x))\, dt\, dx \\ &\leqslant \iint_\Omega \limsup_{n \to -\infty} g(t, x, u_n(t, x))\, dt\, dx \\ &= 0 \end{aligned}$$

which is impossible because $\iint_\Omega h(t, x)\, dt\, dx < 0$. Therefore, we cannot have $\alpha_n \to -\infty$ or $\alpha_n \to +\infty$ so $|\alpha_n| < M_2$ for some $M_2 > 0$.

The above fact, combined with $\| u_{1n} \|_{L^2} \leqslant M$, shows that if (u, λ) is any solution to (3.2) then

$$\begin{aligned} \| u \|_{L^2} &= \| \alpha_n + u_{1n} \|_{L^2} \\ &\leqslant 2\pi\, |\alpha_n| + \| u_{1n} \|_{L^2} \\ &\leqslant 2\pi M_2 + M. \end{aligned}$$

This shows that (a) is true for any ball G_1 in $L_2(\Omega)$, centered at the origin and with radius larger than $\rho_1 = 2\pi M_2 + M$; i.e.,

$$G_1 = \{ u \in L^2(\Omega) : \| u \|_{L^2} \leqslant r,\ r > \rho_1 \}.$$

Now we will show that condition (b) is satisfied. Since J: Im $Q \to$ Ker L is any isomorphism and dim[Im Q] = dim[Ker L] = 1, we may take J to be the identity on $\mathbf{R}$. Now for $\alpha \in \operatorname{Ker} L = \mathbf{R}$

$$\begin{aligned} JQN(\alpha) &= -\frac{1}{4\pi^2} \iint_\Omega [g(t, x, \alpha) + h(t, x)]\, dt\, dx \\ &= -\frac{1}{4\pi^2} \left[\iint_\Omega g(t, x, \alpha)\, dt\, dx + \iint_\Omega h(t, x)\, dt\, dx \right]. \end{aligned}$$

Since $\liminf_{\alpha \to +\infty} g(t, x, \alpha) = \infty$ uniformly on Ω, with Fatou's lemma and (H_4),

$$\liminf_{\alpha \to +\infty} \iint_\Omega g(t, x, \alpha)\, dt\, dx \geqslant \iint_\Omega \liminf_{\alpha \to +\infty} g(t, x, \alpha)\, dt\, dx = \infty.$$

Therefore there exists $\alpha_1 > 0$ such that, for $\alpha > \alpha_1$

$$\iint_\Omega g(t, x, \alpha)\, dt\, dx > -\iint_\Omega h(t, x)\, dt\, dx.$$

Therefore $JQN(\alpha) < 0$ for $\alpha > \alpha_1$.

Now let $\alpha > 0$; then $g(t, x, \alpha) \leqslant \varphi(t, x)$ a.e. on Ω. Thus

$$\begin{aligned} 0 &\leqslant \limsup_{\alpha \to -\infty} \iint_\Omega g(t, x, \alpha)\, dt\, dx \\ &\leqslant \iint_\Omega \limsup_{\alpha \to -\infty} g(t, x, \alpha)\, dt\, dx = 0. \end{aligned}$$

Therefore, since $-\iint_\Omega h(t, x)\, dt\, dx > 0$, there exists $\alpha_2 > 0$ such that $\iint_\Omega g(t, x, \alpha)\, dt\, dx < -\iint_\Omega h(t, x)\, dt\, dx$ for $\alpha < -\alpha_2$. Hence $JQN(\alpha) > 0$ for $\alpha < -\alpha_2$.

Therefore there exists $\alpha_0 > 0$ such that

$$JQN(\alpha) < 0 \qquad \text{and} \qquad JQN(-\alpha) > 0 \qquad \text{for} \quad \alpha > \alpha_0.$$

Now take $\rho_2 > 0$ such that $(1/2\pi)[\rho_2^2 - 2\rho_2 M - M^2]^{1/2} > \alpha_0$ and let G_2 be the ball in $L^2(\Omega)$ with radius ρ_2 centered at origin; i.e.,

$$G_2 = \{u \in L^2(\Omega) \colon \|u\|_{L^2} \leqslant \rho_2\}.$$

Then if $u \in G_2$, then $4\pi^2 |\alpha|^2 = \iint_\Omega |\alpha|^2\, dt\, dx$

$$\begin{aligned} &\geqslant \iint_\Omega [|u(t, x)| - |u_{1n}(t, x)|]^2\, dt\, dx \\ &\geqslant \iint_\Omega |u(t, x)|^2\, dt\, dx - 2 \iint_\Omega |u(t, x)|\, |u_{1n}(t, x)|\, dt\, dx \\ &\quad + \iint_\Omega |u_{1n}(t, x)|^2\, dt\, dx \\ &\geqslant \|u\|_{L^2}^2 - 2\, \|u\|_{L^2}\, \|u\|_{L^2} - \|u_{1n}\|_{L^2}^2 \\ &\geqslant \rho_2^2 - 2\rho_2 M - M^2, \text{ so } |\alpha| > \alpha_0. \end{aligned}$$

Take $\rho = \max\{\rho_1, \rho_2\}$ and let $G = \{u \in L^2(\Omega): \|u\|_{L^2} \leqslant \rho\}$. Then conditions (a), (b) are satisfied and our proof is completed.

Remark 3.1. Our condition $\iint_\Omega h(t, x)\, dt\, dx < 0$ is almost necessary for the existence of a weak solution with our condition on g.

Indeed, if the double-periodic problem of the equation

$$\beta u_t + u_{tt} - u_{xx} - g(t, x, u) = h(t, x),$$

$\beta \neq 0$, and $g(t, x, u) \geqslant 0$ on $\Omega \times \mathbf{R}$ has a weak solution, then u satisfies $Lu + Nu = 0$. By taking the inner product with 1, we have $(Lu, 1) + (Nu, 1) = 0$. Since $(Lu, 1) = 0$, $(\mathrm{Nu}, 1) = 0$ or equivalently

$$-\iint_\Omega g(t, x, u)\, dt\, dx = \iint_\Omega h(t, x)\, dt\, dx \leqslant 0.$$

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